Vacuum energy of two-dimensional $\mathcal{N} = (2, 2)$ super Yang-Mills theory

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We measure the vacuum energy of two-dimensional $\mathcal{N} = (2, 2)$ super Yang-Mills theory using lattice simulation. The obtained vacuum energy density is $\mathcal{E}_0 = 0.09(9)(^{+10}_{-8})g^2$, where the first error is the systematic and the second is the statistical one, measured in the dimensionful gauge coupling g which governs the scale of the system. The result is consistent with unbroken supersymmetry, although we cannot exclude a possible very small non-zero vacuum energy.

PACS numbers: 11.15.Ha, 11.30.Pb, 11.30Qc

I. INTRODUCTION

Lattice simulation of gauge theory is of great importance since it provides a non-perturbative approach. For the supersymmetric systems, both theoretical formulation on the lattice and simulation techniques are now rapidly developing (see Ref. [1] for a review and Refs. [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13] for recent related works). One of the mysterious points of supersymmetry (SUSY) is that it is broken in our universe, although it is widely believed that SUSY is a relevant symmetry beyond the standard model. Because SUSY is such a strong symmetry, it is difficult to break it spontaneously in perturbative calculations, unless it is broken at the tree level. The lattice simulation will help to understand the non-perturbative effects for the SUSY breaking.

In a recent work [7], on the basis of a formulation by Sugino [14] the author together with Suzuki explicitly confirmed that lattice artifacts which break supersymmetry in fact disappear in the continuum limit. The target system was two-dimensional $\mathcal{N} = (2, 2)$ super Yang-Mills theory (for other lattice formulations of this system, see Refs. [15, 16, 17, 18, 19, 20, 21]). In order to suppress the fluctuation along the flat direction, we added scalar mass terms and numerically observed the Partially Conserved SuperCurrent (PCSC) relation, which is the conservation law of the supercurrent modified by the scalar mass. Furthermore, we illustrated some physical applications, power-like behavior of certain correlation functions and static potential between fundamental probe charges [13].

The vacuum energy is the order parameter for SUSY breaking [22] and we are interested in the spontaneous SUSY breaking of this system. Hori and Tong [23] pointed out the possibility of spontaneous breaking while the previous simulation presented in Refs. [4, 5] was consistent with no SUSY breaking. The vacuum energy obtained by the above simulation, however, had a rather large error because of an inefficient simulation algorithm.

The purpose of this letter is to measure the vacuum energy with a small error which would help to determine whether supersymmetry is spontaneously broken or not in this system. We need, of course, to separate the spontaneous supersymmetry breaking from a breaking caused by lattice artifacts. The result in Ref. [7] shows that the latter does not survive in the continuum limit.

The target two-dimensional system is a nice arena for simulation of supersymmetric systems, although it is a toy system from a viewpoint of phenomenologically realistic four dimensional models. Since it is two-dimensional, computational cost is less than four-dimensional case. Furthermore, there is no sign problem in the continuum target theory, which also makes simulation easy. Important differences from four-dimensional $\mathcal{N} = 1$ supersymmetric Yang-Mills [28] are the existence of massless scalar fields and the existence of flat directions in the classical potential.

II. FORMULATION

We use the lattice model in Ref. [14] and the method proposed in Refs [4, 5] to measure the vacuum energy.

The action of the target $\mathcal{N} = (2, 2)$ super Yang-Mills theory is obtained by dimensional reduction from 4-dimensional $\mathcal{N} = 1$ super Yang-Mills theory. In the Euclidean continuum spacetime, it is

$$S = \frac{1}{g^2} \int d^2x \ \mathrm{tr} \left\{ \frac{1}{2} F_{MN} F_{MN} + \Psi^T C \Gamma_M D_M \Psi + \widetilde{H}^2 \right\}. \tag{1}$$

Here, M and N run from 0 to 3. Γ_M are the gamma matrices in four dimensions and C is the charge conjugation matrix. The covariant derivative D_M is $\partial_M + i[A_M, \cdot]$, where $\partial_2 = \partial_3 = 0$, A_0 and A_1 are two-dimensional gauge fields, A_2 and A_3 are real scalar fields. The field strength is $F_{MN} = \partial_M A_N - \partial_N A_M + i[A_M, A_N]$. The fermion $\Psi = (\psi_0, \psi_1, \chi, \eta/2)^T$ is a 4 component spinor and $\widetilde{H}$ is a bosonic auxiliary field. The gauge coupling g has mass dimension 1 and we use it as a unit for dimensionful quantities. The action is invariant under four super transformations and one can rewrite it in a supercharge exact form.

The lattice action we use is $S_{\text{lattice}} = S_{\text{Sugino}} + S_{\text{scalar mass}}$ [7], where S_{Sugino} is the lattice version of (1)

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and has one exact nilpotent symmetry Q at finite lattice spacings out of four supercharges [14]. $S_{\text{scalar mass}}$, which softly breaks the supersymmetry as well as Q-symmetry, is needed to lift the flat direction of the potential, because otherwise the simulation does not give thermalized configurations [25]. We thus *define* the system as an extrapolation in which the scalar mass μ goes to 0 after taking continuum limit. In taking the continuum limit, we use fixed scalar mass in a physical unit.

The vacuum energy is the order parameter for SUSY breaking. It is positive if and only if SUSY is spontaneously broken. Therefore it is of crucial importance to choose the correct origin of the energy. Our choice is to use a Q-exact Hamiltonian (density). In Refs. [4, 5], the author together with his collaborators discussed that this choice is consistent with a property of the Witten index. They also confirmed that this method reproduces known results for supersymmetric quantum mechanics. For the current system, the Hamiltonian density is $\mathcal{H} = \frac{1}{2} Q \mathcal{J}_0^{(0)}$, where $\mathcal{J}_0^{(0)}$ is the 0-th component of the Noether current corresponding to another supercharge Q_0. Note that Q and Q_0 satisfy the super algebra $\{Q, Q_0\} = -2i\partial_0$ in the continuum. Since we do not have a lattice version of the Q_0 transformation, we use a naive discretization of the continuum current (see Ref. [5] for its explicit expression). We do not include the scalar mass term in the Hamiltonian density because the Noether current $\mathcal{J}^{(0)}$ does not contain it. The Hamiltonian density is exactly identical to the one treated in Refs [4, 5].

We utilize the temperature as a conjugate external field to the Hamiltonian, i.e., we impose antiperiodic boundary condition (aPBC) for fermion fields in time direction. We use a finite spacial size L_{S} and temporal size β, where β is the inverse temperature as well. What we measure is

$$\langle \mathcal{H} \rangle = \frac{\int_{\text{aPBC}} \mathrm{d}\mu \, \mathcal{H} \exp(-S_{\text{lattice}})}{\int_{\text{aPBC}} \mathrm{d}\mu \, \exp(-S_{\text{lattice}})}, \tag{2}$$

where we explicitly indicate the boundary condition. $\mathrm{d}\mu$ denotes an integration measure of the path integral. If we adopted the periodic boundary condition, since the expectation value of Hamiltonian would be proportional to a derivative of the Witten index w.r.t the coupling constant, the expectation value would be always zero [29]. Therefore, we could not obtain useful information for the SUSY breaking. We will take the zero temperature limit at the final stage to obtain the vacuum energy density: $\mathcal{E}_0 \equiv \lim_{\beta\to\infty}(\lim_{\mu\to 0}(\lim_{a\to 0}\langle\mathcal{H}\rangle))$ [30].

III. SIMULATION

We evaluate the expectation value (2) using a Monte Carlo simulation. The detail of generating the configurations is found in Refs. [7, 25] [31]. We use the Rational Hybrid Monte Carlo (RHMC) method together with the multi-time step for molecular dynamics. Compared with the method used in [4, 5], which uses quenched configurations and reweighting method to treat the fermion effect, the statistical error is well under control.

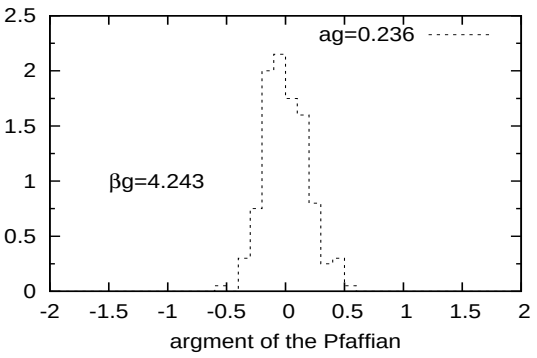

FIG. 1: Distribution of the argument of the Pfaffian for 200 samples. The scalar mass is $\mu = 0.5g$. The argument is centered around 0, which justifies the replacement of the Pfaffian by the positive square root of the determinant.

The effect of the complex phase of the Pfaffian of the Dirac operator is taken into account by phase reweighting of the square root of the determinant. Note that the target system in the continuum has real positive Pfaffian. Because our choice of the lattice spacing is small enough and the argument of the determinants is distributed around zero, there is almost no sign ambiguities caused by taking the square root. In Fig. 1 and Fig. 2, we plot the distribution of the argument of the determinant and a sample distribution from a direction calculation of the Pfaffian [32]. Fig. 1 shows that the Pfaffian is almost real and positive [33] even for the largest lattice spacing and the lowest temperature, and guarantees the replacement of Pfaffian by the positive (i.e., the real part is positive) square root of the determinant. Note that the direct calculation of the Pfaffian is numerically very expensive. From Fig. 2, one can see that as the lattice spacing becomes small the determinant and hence the Pfaffian approaches real and positive values, which is consistent with the property in the continuum.

In the measurement, we calculate the Hamiltonian density every 10 trajectories. We use binning technique (a typical bin size is 5) and use the jack-knife method to estimate the statistical error for each parameters. The continuum limit is taken as a linear extrapolation w.r.t. the lattice spacing a. The massless extrapolation is taken as a linear extrapolation w.r.t. μ^2. We use χ^2 fit and estimate the statistical error of the fitting parameters as the value for which χ^2 increases by 1.

The common parameters are the following. We use SU(2) gauge group. Physical spacial size L_{S} is fixed to $L_{\text{S}} g = 1.414$. The parameter ϵ for the admissibility condition in S_{Sugino} is 2.6.

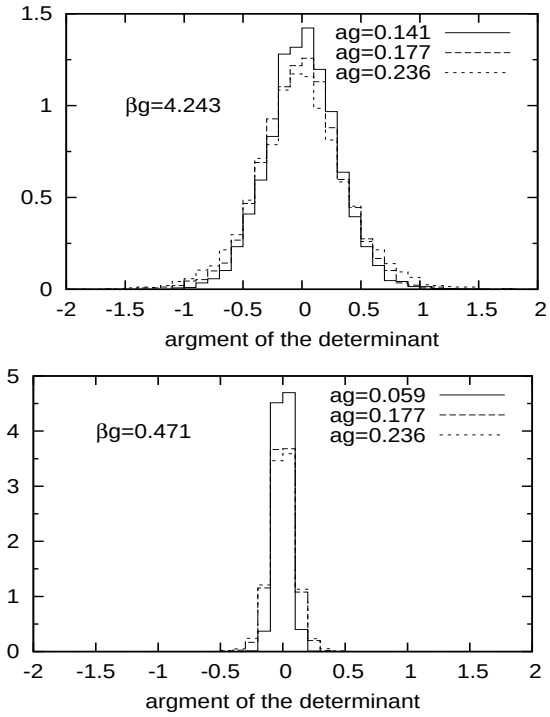

FIG. 2: Distributions of the argument of the determinant. The upper plot is one of the broadest distribution and the lower is one of the narrowest. For each plot, the scalar mass is $\mu = 0.5g$. Plot with $\beta g = 4.243$, $ag = 0.236$ uses the same parameter as the plot in Fig. 1. Even in the broadest cases, the argument is centered around zero.

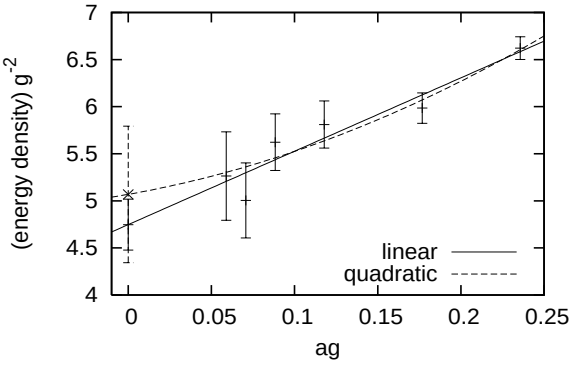

FIG. 3: Example of the continuum limit, for $\beta g = 0.707$ and $\mu = 0.7g$. We use data set with $\beta g = 0.707$ to estimate systematic errors associated with the choice of the fitting function and the fitting region.

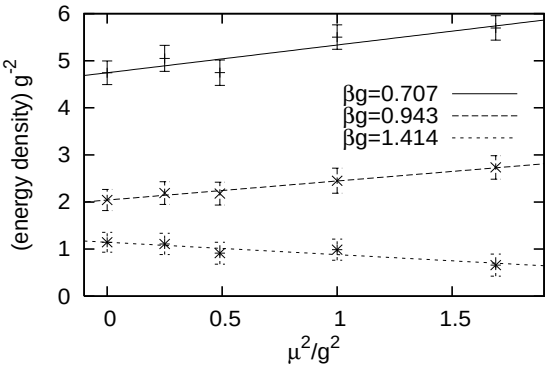

FIG. 4: Examples of the extrapolation of the scalar mass μ to 0. The linear extrapolations in μ^2 fit very well.

IV. RESULTS

We list the detailed parameters and raw results in Table I. For large β, we use 3 different lattice spacings to obtain the continuum limit while for small β we use 4 or more. It is because that in the former cases the obtained values are almost constant within the statistical error over the different lattice spacings; in the latter cases they depend on the lattice spacing.

We estimate a systematic error in taking the continuum limit using the data set with $\beta = 0.707g$ (Fig. 3). Although we adopt the linear fitting to obtain the continuum limit, we tried a quadratic function $A'a^2 + B'a + C'$ as well as a linear one $Aa + B$ in the lattice spacing a. After extrapolating $\mu \to 0$, we obtained two different values and we regard the difference as the systematic error in choosing the fitting function, which is 4.0%. To estimate the errors associated with a choice of the fitting region, we repeated a similar analysis with linear fitting using small 4 lattice spacings and large 3 lattice spacings. Comparing these two results with the one obtained from all 6 lattice spacings, we estimate the error associated with the choice of the fitting region at 6.8%. In total, the systematic error in taking the continuum limit is 11%.

After taking the continuum limit, we extrapolate the scalar mass $\mu \to 0$ (Fig. 4). As the figure indicates, a possible systematic error in the extrapolation is small compared with the statistical ones and thus is negligible.

The final result is given in Fig. 5. We have fitted the result using two different functions: a power of β, $\mathcal{E}(\beta)/g^2 = a_0(\beta g)^{-a_1} + a_2$, and an exponential of β, $\mathcal{E}(\beta)/g^2 = b_0 \exp(-b_1\beta g) + b_2$. We obtain $a_0 = 1.94^{+11}_{-16}$, $a_1 = 2.59^{+9}_{-6}$, $a_2 = 0.09^{+10}_{-8}$ and $b_0 = 157^{+13}_{-12}$, $b_1 = 5.01^{+18}_{-17}$, $b_2 = 0.37^{+8}_{-8}$. The errors are only statistical. The values of χ^2 per degrees of freedom are 1.21 and 3.81 for 7 degrees of freedom, respectively. Using the result of the fit by power function, we obtain $\mathcal{E}_0 = 0.09^{+10}_{-8}g^2$ for the central value and the statistical errors.

However, because the above fits use small β as well as large β, one cannot exclude possible effects from the excited states. In fact, using only large β ($\beta g > 0.9$), we obtain reliable fits for both exponential and power, which gives $\mathcal{E}_0 = 0.17^{+10}_{-11}g^2$ and $\mathcal{E}_0 = 0.05^{+17}_{-17}g^2$, respectively [34]. These values are consistent with the above central value within the statistical error. Using these values, we estimate that the systematic error in taking zero temperature limit is $0.08g^2$.

What is the physical difference between the power and

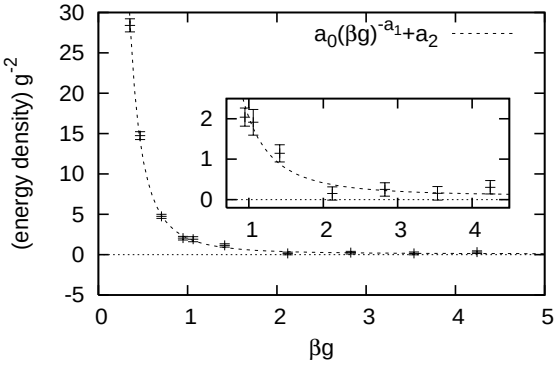

FIG. 5: Energy density $\mathcal{E}(\beta)$. Errors bars denote statistical errors only. The dashed curve is the best fit of data with a power function of β. As the inverse temperature β becomes large, the energy density approaches 0.

the exponential fitting? If the spectrum has a gap, the energy behaves as an exponential function of β. A power behavior of β comes from gap-less excitations [35]. In a finite volume system, usually we have a discrete spectrum which leads to a mass gap. However, the current system has the *flat direction* in the potential at least in the classical level, and the fluctuation along the flat direction can provide a continuum spectrum. The current result does not allow us to determine which is the case [36]. If the fit by the power was preferred, it would be an evidence of the recovery of the flat direction after the $\mu \to 0$ extrapolation.

In total, we obtain the vacuum energy $\mathcal{E}_0 = 0.09(9)(^{+10}_{-8})g^2$, where the first error is the systematic and the second is the statistical one. It is consistent with 0 within the error, that is, it is consistent with supersymmetry not being spontaneously broken.

V. CONCLUSION

In this letter, we measured the vacuum energy of two-dimensional $\mathcal{N} = (2, 2)$ super Yang-Mills theory using lattice simulation. Because our simulation does not have explicit supersymmetry breakings caused by lattice artifacts, the vacuum energy we measured works as an order parameter of *spontaneous* SUSY breaking. The result is consistent with 0 within the error, which seems to imply that supersymmetry is not spontaneously broken, although we cannot exclude the possibility of a non-zero value smaller than the error which would indicate a spontaneous breaking of SUSY.

Acknowledgments

I. K. is supported by the Special Postdoctoral Research Program at RIKEN. He is also supported by the Nishina Memorial Foundation. He thanks H. Suzuki for encouragement and discussion. He also thanks A. D'Adda, A. Fotopoulos, M. Hanada, D. Kadoh, N. Kawamoto, Y. Kikukawa and F. Sugino for various comments and discussion. The simulation was performed on the RIKEN Super Combined Cluster.

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TABLE I: Expectation value of the Hamiltonian density $\langle\mathcal{H}\rangle$ and number of the independent configurations after binning for each parameters. g is the dimensionful gauge coupling.

			$\mu = 0.5g$		$\mu = 0.7g$		$\mu = 1.0g$		$\mu = 1.3g$	
βg	ag	lat. size	num.	$\langle\mathcal{H}\rangle g^{-2}$	num.	$\langle\mathcal{H}\rangle g^{-2}$	num.	$\langle\mathcal{H}\rangle g^{-2}$	num.	$\langle\mathcal{H}\rangle g^{-2}$
0.354	0.118	3×12	800	34.50(32)	800	34.39(32)	2000	33.57(31)	800	33.55(34)
	0.088	4×16	1200	32.66(35)	1200	32.45(35)	3000	32.49(34)	3000	31.59(35)
	0.071	5×20	1600	31.98(39)	1600	31.99(38)	2000	30.89(38)	800	30.60(40)
	0.059	6×24	400	31.47(66)	800	30.54(65)	400	29.80(67)	800	29.72(65)
0.471	0.157	3×9	800	17.59(24)	800	17.89(22)	800	17.04(21)	1000	17.62(21)
	0.118	4×12	1000	17.06(28)	400	16.66(30)	800	16.58(28)	1000	16.28(28)
	0.079	6×18	800	16.03(32)	1600	16.15(30)	1600	16.31(30)	1600	15.83(30)
	0.059	8×24	2000	16.05(53)	2000	15.64(56)	2000	15.75(56)	500	16.49(58)
0.707	0.236	3×6	400	6.18(12)	800	6.62(12)	800	6.64(11)	800	6.51(11)
	0.177	4×8	200	5.77(17)	800	5.99(16)	800	6.40(15)	800	6.18(16)
	0.118	6×12	800	5.70(23)	500	5.81(24)	2000	5.87(22)	400	5.85(23)
	0.088	8×16	400	5.43(32)	800	5.62(30)	800	5.89(30)	400	6.15(29)
	0.071	10×20	400	4.87(41)	800	5.00(39)	400	5.87(38)	400	5.94(37)
	0.059	12×24	800	5.95(45)	400	5.26(46)	400	6.27(46)	2000	6.36(45)
0.943	0.157	6×9	800	2.62(15)	800	2.65(15)	800	2.93(15)	800	2.96(15)
	0.118	8×12	800	2.54(19)	800	2.73(19)	800	2.75(19)	800	2.61(20)
	0.079	12×18	1600	2.07(21)	1600	2.35(22)	1600	2.62(21)	800	2.72(22)
1.061	0.177	6×8	800	1.70(12)	400	1.97(13)	800	2.09(12)	400	1.97(12)
	0.118	9×12	2000	1.74(17)	800	2.04(18)	2000	2.05(18)	800	2.46(17)
	0.088	12×16	800	1.94(18)	1000	1.85(18)	800	1.82(18)	1600	1.75(17)
1.414	0.236	6×6	800	0.777(76)	400	0.918(81)	800	0.961(77)	2000	1.046(75)
	0.177	8×8	800	1.00(10)	400	1.07(10)	1000	1.08(10)	800	0.97(10)
	0.118	12×12	1600	0.89(11)	800	0.86(11)	1600	0.93(11)	800	0.84(11)
2.121	0.236	9×6	800	0.471(66)	2000	0.542(63)	2000	0.454(61)	2000	0.435(60)
	0.177	12×8	2000	0.377(82)	800	0.574(88)	800	0.491(86)	2000	0.548(82)
	0.118	18×12	800	0.442(93)	1333	0.319(94)	1500	0.55(10)	1200	0.52(10)
2.828	0.236	12×6	800	0.349(57)	2000	0.417(53)	2000	0.442(54)	4000	0.385(52)
	0.177	16×8	800	0.339(78)	800	0.477(75)	2000	0.280(74)	800	0.471(75)
	0.118	24×12	1200	0.349(94)	1500	0.362(90)	300	0.22(10)	1500	0.412(94)
3.536	0.236	15×6	800	0.410(49)	800	0.246(48)	1000	0.326(49)	800	0.370(49)
	0.177	20×8	400	0.359(68)	500	0.366(68)	800	0.346(72)	400	0.379(72)
	0.118	30×12	2000	0.28(10)	1000	0.20(10)	1000	0.300(97)	1000	0.439(98)
4.243	0.236	18×6	2000	0.286(44)	800	0.255(44)	4000	0.302(42)	400	0.346(43)
	0.177	24×8	800	0.290(62)	400	0.307(68)	2000	0.310(60)	800	0.292(64)
	0.141	30×10	800	0.241(80)	800	0.278(78)	400	0.382(79)	800	0.339(78)

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[28] See, for example, Ref. [24] for a recent report and references there in.

[29] In the case of periodic boundary condition, if SUSY is broken, the partition function becomes zero and the expectation value will be $\frac{0}{0}$. Numerically, one will encounter a severe sign problem caused by sign (or complex phase) of the Pfaffian.

[30] In the extrapolation $\mu \to 0$, we use the values at $\mu = 1.3g,\ 1.0g,\ 0.7g$ and $0.5g$ for all β. The Hamiltonian density is smoothly extrapolated for each β. See Fig. 4 and related descriptions.

[31] The code was developed using FermiQCD/MDP [26] and the parameters for rational expansion were obtained from a program in Ref. [27].

[32] The author thanks to H. Suzuki, for letting the author use his code for calculating the Pfaffian.

[33] Although this results is quite reasonable since it is con-

sistent with the real positiveness of the Pfaffian of the continuum theory, it is rather different from the result in Ref. [11]. Important differences from Ref. [11] are lattice model used in the simulation, boundary condition and system size. One (or more) of them may explain the difference.

[34] Other coefficients are $a_0 = 1.9^{+2}_{-2}$, $a_1 = 2.2^{+9}_{-5}$ and $b_0 = 13^{+9}_{-5}$, $b_1 = 2.0^{+6}_{-4}$.

[35] If the energy density function behaves as $\rho(\mathcal{E}) \sim \mathcal{E}^{\nu-1}$, $\mathcal{E}(\beta)$ behaves as ν/β for large β which gives $a_1 = 1.0$.

[36] Although $a_1 = 2.2^{+9}_{-5}$ for $\beta g > 0.9$ is rather different from 1.0 and seems to prefer the exponential fit, we cannot totally exclude the possibility of power function with the current error.